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A COCYCLE CONSTRUCTION FOR GROUPS

BY
HARTMUT LAUE

ABSTRACT
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In the spirit of Wielandt's method of introducing the transfer mapping (see e.g. [5]), we give a cocycle construction with several applications. In particular, our cocycles enable us to construct certain group automorphisms under fairly general conditions.

Let M be a set and H a group acting faithfully on M (i.e. we have a monomorphism of H into S_M). Choose $H_0 \trianglelefteq H$ such that

$$C_H(m) \leqq H_0 \qquad \text{for all } m \in M.$$

(In all our applications, H will act "freely" on M, i.e. $C_H(m) = 1$ for all $m \in M$.) Write $\sim$ for the equivalence relation on M induced by the H-orbits in M. In the following let m, n, r, $s \in M$. For $n \sim m$, the unique coset $H_0h \in H/H_0$ with the property $n = mh$ is denoted by $\frac{n}{m}$. We then have

$$(1) \qquad m \sim n \Rightarrow \frac{n}{m} = \left(\frac{m}{n}\right)^{-1},$$

$$(2) \qquad r \sim m \sim n \Rightarrow \frac{m}{r} \cdot \frac{n}{m} = \frac{n}{r}.$$

Now let G be a group acting on M in such a way that the monomorphic images of H and H_0 in S_M are normalized by the image of G in S_M. Thus a natural action of G on H and H/H_0 is induced (written exponentially). For x, $y \in G$, one easily verifies

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(3) $$m \sim n \Rightarrow mx \sim nx, \qquad \frac{nx}{mx} = \left(\frac{n}{m}\right)^x,$$

(4) $$mx \sim n, \quad ny \sim r \Rightarrow mxy \sim r, \qquad \frac{mxy}{r} = \frac{ny}{r} \cdot \left(\frac{mx}{n}\right)^y,$$

(5) $$r \sim m, \quad mx \sim n, \quad n \sim s \Rightarrow rx \sim s, \qquad \frac{rx}{s} = \left(\frac{s}{n}\right)^{-1} \cdot \frac{mx}{n} \cdot \left(\frac{r}{m}\right)^x.$$

We now suppose there are only finitely many H-orbits in M, and H/H_0 is abelian. Then if R, S are full sets of representatives of the H-orbits in M, we put

$$\frac{S}{R} := \prod_{\substack{r \in R, s \in S \\ r \sim s}} \frac{s}{r} \in H/H_0.$$

Observing that the action of G on M induces an action on the set of full sets of representatives of the H-orbits in M, we define a mapping w_R of G into H/H_0 by

(6) $$w_R(x) := \frac{Rx}{R} \qquad \text{for all } x \in G.$$

THEOREM 1. *Let G, H be groups acting on a set M. Assume that H acts faithfully and there are only finitely many H-orbits in M. Moreover let*

$$\langle C_H(m) \mid m \in M \rangle \cdot H' \leqq H_0 \leqq H$$

and assume the image of G in S_M normalizes the images of H and H_0 in S_M. Let R, S be full sets of representatives of the H-orbits in M. Then

(i) *w_R is a cocycle of G into H/H_0*
 (i.e. $w_R(xy) = w_R(x)^y w_R(y)$ for all x, $y \in G$),

(ii) $w_S(x) = w_R(x) \cdot \prod_{\substack{r \in R, s \in S \\ r \sim s}} \left[\frac{s}{r}, x\right]$ *for all $x \in G$,*

(iii) *if $[G, H] \leqq H_0$, then $w_R = w_S$, and w_R is a homomorphism.*

PROOF. Let x, $y \in G$. For each $r \in R$ there exists exactly one $r^* \in R$ such that $r^* y \sim r$ and furthermore exactly one $r^{**} \in R$ such that $r^{**}x \sim r^*$. By (4), r^{**} is the unique element of R such that $r^{**}xy \sim r$, and we have

$$\frac{r^{**}xy}{r} = \frac{r^*y}{r} \cdot \left(\frac{r^{**}x}{r^*}\right)^y.$$

If r runs through all of R, then so do the corresponding elements r^* and r^{**}. Therefore, using the hypothesis that H/H_0 is abelian,

$$\begin{aligned} w_R(xy) &= \prod_{r\in R} \frac{r^{**}xy}{r} \\ &= \prod_{r\in R} \left(\left(\frac{r^{**}x}{r^*} \right)^y \cdot \frac{r^*y}{r} \right) \\ &= \prod_{r\in R} \left(\frac{r^{**}x}{r^*} \right)^y \cdot \prod_{r\in R} \frac{r^*y}{r} \\ &= w_R(x)^y \cdot w_R(y), \end{aligned}$$

proving (i). Similarly, we prove (ii): For each $s \in S$ there exists exactly one $r \in R$ such that $r \sim s$ and furthermore exactly one $r^* \in R$ such that $r^*x \sim r$. If s^* denotes the unique element of S such that $s^* \sim r^*$, then, by (5), s^* is the unique element of S such that $s^*x \sim s$, and we have

$$\frac{s^*x}{s} = \left(\frac{s}{r}\right)^{-1} \cdot \frac{r^*x}{r} \cdot \left(\frac{s^*}{r^*}\right)^x.$$

If s runs through all of S, then so does s^*, and the corresponding elements r, r^* run through all of R. Therefore

$$\begin{aligned} w_S(x) &= \prod_{s\in S} \frac{s^*x}{s} \\ &= \prod_{s\in S} \left(\left(\frac{s}{r}\right)^{-1} \cdot \frac{r^*x}{r} \cdot \left(\frac{s^*}{r^*}\right)^x \right) \\ &= \prod_{r\in R} \frac{r^*x}{r} \cdot \prod_{\substack{r\in R, s\in S \\ r\sim s}} \left(\frac{s}{r}\right)^{-1} \cdot \prod_{\substack{r\in R, s\in S \\ r\sim s}} \left(\frac{s}{r}\right)^x \\ &= w_R(x) \cdot \prod_{\substack{r\in R, s\in S \\ r\sim s}} \left[\frac{s}{r}, x\right], \end{aligned}$$

as desired. The first part of (iii) is a consequence of (ii), and the second part follows from (i).

We observe that Theorem 1 is equally true if the action of H on M is given by an anti-monomorphism instead of a monomorphism of H into S_M, since H/H_0 is abelian. This remark is useful for our following examples; in the first two the hypothesis of (iii) is fulfilled:

EXAMPLE 1. (The transfer.) Let G be a group, $H' \leqq H_0 \leqq H \leqq G$ and $|G:H|$ finite. We put $M := G$ and let G act by right multiplication, H by left multiplication on $M(=G)$. Then $C_H(m) = 1$ for all $m \in M$, and the H-orbits in M are the right cosets of H in G. We therefore have (see [5], chapter 10)

$$w_R(x) = v_{G\to H/H_0}(x) \qquad \text{for all } x \in G.$$

A second application of Theorem 1 yields the so-called "Totalverlagerung" (total transfer) of [4] which was the key to Berger's description of the smallest normal Fitting class [1]:

EXAMPLE 2. (Laue, Lausch, Pain [4]) Let G be a group, $U \leqq G$, and suppose there are only finitely many subgroups of G isomorphic to U. Let M be the set of all monomorphisms of U into G, and τ_x the inner automorphism of G induced by $x \in G$. Then for each $\varphi \in M$ we have $\varphi\tau_x \in M$. This gives a natural action of G on M. Moreover, the group $H := \operatorname{Aut} U$ acts faithfully on M by left multiplication, since for $\varphi \in M$, $\alpha \in H$, we have $\alpha\varphi \in M$. Now suppose $H' \leqq H_0 \leqq H$, choose a full set R of representatives of the H-orbits in M, and denote by f^{H_0} the total transfer for the subgroup U of G (with respect to G, see [4]). Then

$$w_R(x) = f^{H_0}(x) \qquad \text{for all } x \in G.$$

For if we choose for each subgroup $V \cong U$ an isomorphism φ_V of U onto V, then $R := \{\varphi_V \mid V \leqq G,\ V \simeq U\}$ is a full set of representatives of the H-orbits in M, and for all $x \in G$ we have

$$f^{H_0}(x) = H_0 \cdot \prod_V \varphi_V\tau_{x|V}\varphi_{V^x}^{-1} = \prod_V \frac{\varphi_V x}{\varphi_{V^x}} = \frac{Rx}{R} = w_R(x).$$

So far the applications of Theorem 1 have led to homomorphisms, in both cases due to the fact that H acted by left multiplication, G by right multiplication on M, whence the induced permutation groups centralized each other.

EXAMPLE 3. (Brandis [2]) Let X be a group, $H \leqq X$, $G \leqq N_X(H)$ and M a G-invariant non-empty subset of X which is the union of a finite set of right cosets of H in X (e.g. $H \leqq M \trianglelefteq X$, $|M:H|$ finite). Let G act by conjugation, H by left multiplication on M. We have $(hm)^g = h^g m^g$ for $h \in H$, $g \in G$, $m \in M$, and the H-orbits in M are the right cosets of H in X which are contained in M. Let R be a full set of representatives of these and $H' \leqq H_0 \leqq H$, H_0 invariant under G. Then

$$w_R(g) = \prod_{\substack{m,n \in R \\ Hn^g = Hm}} H_0 n^g m^{-1} \qquad \text{for all } g \in G \tag{7}$$

and consequently, in the case of a subgroup M of X such that $|M:H|$ is finite,

$$w_R(h) = H_0 h^{-|M:H|} v_{M\to H/H_0}(h) \qquad \text{for all } h \in H \cap G. \tag{8}$$

Let us now consider the important case that H is a normal subgroup of G, and the action of H on M coincides with the action of G, restricted to H:

EXAMPLE 4. (Brandis [2]) Let G be a group, $H \trianglelefteq G$, and $U \leqq G$ such that $U \cap H = 1$ and $|G : HU|$ is finite. Let M be the set of right cosets of U in G, and define the action of G (and H) on M by right multiplication. Then $C_H(m) = 1$ for all $m \in M$. If R_0 is a full set of right coset representatives of HU in G, then $R := \{Ur \mid r \in R_0\}$ is a full set of representatives of the H-orbits in M. For $g \in G$ and $r \in R_0$ there is a unique $u_{r,g} \in U$ such that $u_{r,g} rg \in R_0 H$, and we have $u_{r,h} = 1$ for all $h \in H$. Suppose $H' \leqq H_0 \leqq H$. Then

$$w_R(g) = \prod_{\substack{r,s \in R_0 \\ HUrg = HUs}} H_0 s^{-1} u_{r,g} rg \qquad \text{for all } g \in G, \tag{9}$$

and consequently

$$w_R(h) = H_0 h^{|G:HU|} \qquad \text{for all } h \in H. \tag{10}$$

The cocycles in Examples 3 and 4 can be calculated fairly well for elements of $H \cap G$ resp. H. In particular, under convenient order assumptions, one can conclude $\ker w_R \cap H = H_0$. Using the simple fact that the kernel of a cocycle is a subgroup, Brandis [2] gave elegant proofs for important splitting theorems for finite groups along these lines. Thus we immediately arrive at the well-known generalization of the Schur–Zassenhaus theorem by Gaschütz [3] if we suppose $(|G : HU|, |H/H_0|) = 1$ in Example 4.

We shall now apply Theorem 1 to construct group automorphisms which centralize large factor groups. The following two results are typical:

THEOREM 2. *Let G be a group, A an abelian normal subgroup of G, $A \leqq B \leqq G$, $|G : B|$ finite, and assume B splits over A. Let t be an integer such that $a \mapsto a^{|G:B|t+1}$ is an automorphism of A. Then there is an automorphism α of G such that $[G, \alpha] \leqq A$ and $a^\alpha = a^{|G:B|t+1}$ for all $a \in A$.*

THEOREM 3. *Let G be a group, A an abelian normal subgroup of G, $A \leqq M \trianglelefteq G$, $|M : A|$ finite, and let t be an integer such that $aC_A(M) \mapsto a^{1-|M:A|t} C_A(M)$ is an automorphism of $A/C_A(M)$. Then there is an automorphism α of G such that $[G, \alpha] \leqq A$ and*

$$a^\alpha = a^{1-|M:A|t}\, \mathrm{Tr}_{M/A}(a)^t \qquad \textit{for all } a \in A.$$

(Here $\mathrm{Tr}_{M/A}(a) := \prod_{Am \in M/A} a^m$.)

Of course, in the finite case the mapping $a \mapsto a^{|G:B|t+1}$ (resp. $aC_A(M) \mapsto a^{1-|M:A|t}C_A(M)$) is an automorphism of A (resp. $A/C_A(M)$) if and only if $(|A|, |G:B|t+1)=1$ (resp. $(|A/C_A(M)|, |M:A|t-1)=1$). The following lemma connects group endomorphisms and the cocycles of Theorem 1. In particular, by means of the cocycles of Examples 3 and 4, this yields proofs of Theorems 2 and 3.

LEMMA. *Let G be a group and A an abelian normal subgroup of G.*

(i) *Let w be a cocycle of G into A and put $g^\alpha := g \cdot w(g)$ for all $g \in G$. Then α is an endomorphism of G, and* $\ker \alpha \subseteq A$.

(ii) *Let M be a set, $\bar{\ }$ a homomorphism of G and $\tilde{\ }$ a G-monomorphism of A into S_M. Suppose there are only finitely many $\tilde{A}$-orbits in M and $C_{\tilde{A}}(m)=1$ for all $m \in M$. Let R be a full set of representatives of the $\tilde{A}$-orbits in M and define w_R as in* (6). *Then $w := w_R$ satisfies the hypothesis of* (i).

PROOF. (i) is trivial. As for (ii), we apply Theorem 1(i), putting $H := A$, $H_0 := 1$. As $\tilde{\ }$ is a G-monomorphism, the action of G on A induced by the action of $\bar{G}$ on $\tilde{A}$ is simply given by the usual conjugation within G.

PROOF OF THEOREM 2. In order to establish the situation of Example 4, we put $H := A$, $H_0 := 1$, choose a complement U of A in B, and let both G and A act by right multiplication on the set M of right cosets of U in G. Define w_R as in (9), and put $w(g) := w_R(g)^t$ for all $g \in G$. Then we can apply our Lemma, part (i) of which gives us an endomorphism α of G such that $\ker \alpha \subseteq A$. But for $a \in A$ we have, by (10), $a^\alpha = a \cdot a^{|G:AU|t} = a^{|G:B|t+1}$. Therefore $\ker \alpha = 1$, and α is an automorphism.

PROOF OF THEOREM 3. We proceed similarly, but apply the cocycle construction of Example 3. We put $X := G$, $H := A$, $H_0 := 1$, thus arriving at a situation which has been dealt with in Example 3. We define w_R as in (7), put $w(g) := w_R(g)^t$ for all $g \in G$, and observe that our Lemma can be applied, yielding an endomorphism α of G such that $\ker \alpha \subseteq A$. Since $A \trianglelefteq M$, we have $v_{M \to A}(a) = \mathrm{Tr}_{M/A}(a)$ for all $a \in A$. Now if $a \in A$ and $a^\alpha = 1$, then by (8), $a^{|M:A|t-1} = \mathrm{Tr}_{M/A}(a)^t \in C_A(M)$ which implies $a \in C_A(M)$, according to our hypothesis. But this yields $\mathrm{Tr}_{M/A}(a) = a^{|M:A|}$, hence $a^{|M:A|t} = a^{|M:A|t-1}$, and therefore $a = 1$.

We give a further example of group automorphisms constructed by means of our Lemma: Let G be a group, A a finite abelian normal subgroup of G, $n \leqq |A|$ and M the set of all those subsets of n elements of A which cannot be obtained as the union of a set of cosets of a cyclic subgroup $\neq 1$ of A. (In the case $(n, |A|) = 1$, M consists of all subsets of n elements of A.) Let G act on M by conjugation, A by multiplication. If we have $T \in M$, $a \in A$ and $Ta = T$, then $T = \bigcup_{t \in T} t\langle a \rangle$, hence $a = 1$. Therefore $C_A(T) = 1$ for all $T \in M$. We choose a full set R of representatives of the A-orbits in M, and (having observed the trivial equality $Ta^g = (T^{g^{-1}}a)^g$ for $T \in M$, $a \in A$, $g \in G$) apply our Lemma. As $w(g) := w_R(g) = 1$ for all $g \in C_G(A)$, we see that the resulting endomorphism is an automorphism of G which centralizes G/A and $C_G(A)$. For $n = 1$ we thus get exactly the inner automorphisms of G which are induced by elements of A. On the other hand, if G is a nonabelian group of order 8, A cyclic of order 4, $n = 2$, then

$$|M| = \binom{4}{2} - 2 = |A|,$$

and M therefore consists of a single A-orbit; an easy calculation shows that our resulting automorphism is non-inner in this case.

In general, it seems hard to decide if an automorphism constructed by means of our Lemma is inner or even a transformation by an element of A. One expects the answers to these questions do not depend on the choice of R, as will indeed be proved now:

THEOREM 4. *In the situation of the Lemma, part* (ii), *let* R, S *be full sets of representatives of the* $\tilde{A}$*-orbits in* M *and* α_R, α_S *the endomorphisms of* G *corresponding to* $w = w_R$, *resp.* $w = w_S$ *via part* (i) *of the Lemma. If* α_R *is an automorphism, then so is* α_S, *and in* Aut G, *the elements* α_R, α_S *represent the same left coset of the subgroup of inner automorphisms induced by elements of* A.

PROOF. We first note

(11) If $a \in A$, $\alpha \in \operatorname{Aut} G$, $[G, \alpha] \leqq A$ and the mapping γ is defined by $x^\gamma := x^\alpha [a, x]$ for all $x \in G$, then $a^{\alpha^{-1}\gamma} = x^{a^{-1}}$ for all $x \in G$.

Now if $\mathfrak{a}$ is a mapping of a finite set, say $\{1, \ldots, k\}$, into A, then we conclude inductively from (11): The mapping

$$\gamma : G \to G, \qquad x \mapsto x^\alpha \prod_{j \leq k} [\mathfrak{a}(j), x]$$

is an automorphism of G, and $\alpha^{-1}\gamma$ is induced by an element of A. As Theorem 1 (ii) and the fact that $\tilde{}$ is a G-homomorphism yield

$$x^{\alpha_S} = x \cdot w_S(x) = x \cdot w_R(x) \cdot \prod_{\substack{r \in R, s \in S \\ r \sim s}} \left[\frac{s}{r}, x\right] = x^{\alpha_R} \cdot \prod_{\substack{r \in R, s \in S \\ r \sim s}} \left[\frac{s}{r}, x\right],$$

the theorem follows.

We finally show that any application of Theorem 1 can be viewed as a subcase of Example 4, which is therefore in a certain sense a frame of all our cocycle constructions.

THEOREM 5. *Under the hypotheses of Theorem* 1, *put* $A := H/H_0$ *and denote by* X *the semidirect product of* G *and* A (*the operation given, as before, by the embeddings of* G *and* H *in* S_M). *Then there are* (*not necessarily distinct*) *subgroups* $U_1, \ldots, U_n$ *of* X *which intersect* A *trivially, and for each* j $(1 \leqq j \leqq n)$ *a full set* R_j *of right coset representatives of* U_j *in* X *such that, writing* w_j *for the corresponding cocycle given by* (8), *we have*

$$w_R(x) = \prod_{j=1}^{n} w_j(x) \qquad \text{for all } x \in G.$$

PROOF. Let Y be the semidirect product of G and H. Then there is a natural action of Y on M which may w.l.o.g. be assumed transitive, as a glance at the definition of w_R ensures. We choose $m \in M$ and put $V := C_Y(m)$. Then $V \cap H = C_H(m) \leqq H_0$. Y acts by right multiplication on the set Y/V of right cosets of V in Y, and the mapping

$$\varphi : M \to Y/V, \qquad my \mapsto Vy$$

is an isomorphism of Y-spaces, transforming R into a full set of representatives of the H-orbits in Y/V, i.e. the right cosets of VH in Y. Writing ψ for the canonical epimorphism of Y onto X, we identify G and G^ψ and put $U := V^\psi$. Then $R^{\varphi\psi}$ is a full set of representatives of the A-orbits in the set $(Y/V)^\psi$ of right cosets of U in X, and for all $x \in G$, $r, s \in R$ such that $rx \sim s$ we have

$$(r^\varphi)x \sim s^\varphi, \quad r^{\varphi\psi}x^\psi \sim s^{\varphi\psi}, \quad \text{and} \quad \frac{rx}{s} = \frac{r^\varphi x}{s^\varphi} = \frac{r^{\varphi\psi}x}{s^{\varphi\psi}}.$$

This implies $w_R = w_{R^{\varphi\psi}}$, as desired. The proof shows that the number n in Theorem 5 can be chosen as the number of X-orbits in M.

In this paper we aimed at a unifying treatment (Theorem 1) for several kinds of what is sometimes called the "averaging process" in group theory, the transfer being a prominent example of the latter. Calculating kernels has proved a strong and successful method in numerous proofs using transfer homomorphisms, but has moreover led to remarkable results in cases of nonhomomorphic cocycle constructions as well (Examples 3 and 4, see [2]), and certainly deserves further attention. Rather than an analogy between the cocycles of Theorem 1 and the transfer we have in fact a specialization (Example 1), to the effect that the kernel specializes from a mere subgroup to a normal one. In a different direction, we have taken the opportunity to point out possibilities of getting automorphisms from the cocycle construction. The latter is of "purely group theoretic" interest to us, and we think nothing essential for our modest purposes would have been gained, say, by highbrow cohomological vocabulary, from which we have refrained.

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MATHEMATISCHES SEMINAR DER UNIVERSITÄT
OLSHAUSENSTR. 40
D 2300 KIEL 1, W GERMANY